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AN ELASTICITY SOLUTION FOR TRANSVERSELY INEXTENSIBLE CIRCULAR CYLINDRICAL SHELLS

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Abstract—The problem of transversely inextensible, circular cylindrical shells subjected to arbitrarily prescribed edge-stresses or displacements is considered. Solutions are sought in terms eigenfunctions which decay exponentially in the axial direction. The edge-conditions are treated using a biorthogonality property of these eigenfunctions. Numerical examples are presented comparing the present exact results with the known asymptotic results.

INTRODUCTION

In a previous paper[1] we considered the problem of transversely inextensible, semi-infinite circular cylindrical shells subjected to arbitrarily prescribed end stresses or end displacements. Assuming the thickness to diameter ratio of the shell to be small, an asymptotic approach was used to obtain the "interior" as well as the "St. Venant edge-zone" forms of the equations of elasticity and also to show how to obtain the appropriate boundary conditions for these two asymptotic sequences of differential equations. While the asymptotic analysis relates three dimensional elasticity theory to two dimensional shell theories, the elasticity problem posed in[1] also admits an exact infinite series solution. The present paper deals with this solution. Through the use of biorthogonal eigenfunctions it is shown that all elastically admissible combinations of stress and displacement edge-conditions can be treated without resorting to the solution of an infinite system of equations—a situation occurring in the problem of end-loaded isotropic cylinders[2]. We attempt to develop the solution with the specific purpose of verifying the results of[1] insofar as thin shells are concerned. The "interior" and the "edge-zone" contributions of the asymptotic theory appear in a recognizable form in terms of the eigenfunctions. The characteristic exponents determining the axial decay of the solutions comeout to be remarkably close to those obtained in[1]. Numerical examples are presented to illustrate the effects of thickness to diameter ratio and of shear deformability.

FORMULATION OF THE PROBLEM [1]

The differential equations for symmetric deformations of transversely inextensible shells consist of the equilibrium equations

$$r\sigma_{x,x} + (r\tau)_{,r} = 0, \quad r\tau_{,x} + (r\sigma_r)_{,r} - \sigma_\theta = 0, \tag{1}$$

and the constitutive relations

$$u_{,x} = \frac{\sigma_x - \nu\sigma_\theta}{E}, \quad \frac{v}{r} = \frac{\sigma_\theta - \nu\sigma_x}{E},$$
$$v_{,r} = 0, \quad u_{,r} + v_{,x} = \frac{\tau}{G}, \tag{2}$$

where E, G and ν are independent constants.

The system of eqns (1) and (2) is to be solved in the region $0 < x < \infty$, $a < r < b$ subject to the surface conditions

$$\sigma_r(a, x) = \tau(a, x) = 0, \quad \sigma_r(b, x) = \tau(b, x) = 0 \tag{3}$$

and subject to any one of the following sets of conditions for $x = 0$ and $a \le r \le b$

$$\sigma_x = \bar{\sigma}_x, \quad v = V^*, \tag{3a}$$

$$\sigma_x = \bar{\sigma}_x, \quad Q = Q^*, \tag{3b}$$

$$u_{,r} = \bar{u}_{,r}, \quad v = V^*, \tag{3c}$$

$$u_{,r} = \bar{u}_{,r}, \quad Q = Q^*, \tag{3d}$$

where $\bar{\sigma}_x$ and $\bar{u}_{,r}$ are prescribed functions of r and V^* and Q^* are prescribed constants. As shown in[1], in the case of transversely inextensible shells v is independent of r and the appropriate edge condition, instead of a prescribed shear stress is a prescribed transverse shear stress resultant given by

$$Q^* = \int_a^b \frac{2\tau r}{a+b} \, \mathrm{d}r. \tag{4}$$

We further stipulate that the stresses vanish as $x \to \infty$. For this, $\bar{\sigma}_x$ must be such that $\int_a^b \bar{\sigma}_x r \, \mathrm{d}r = 0$.

From eqn (2) we obtain

$$v = V(x), \qquad u_{,r} = \frac{\tau}{G} - V_{,x} \tag{5}$$

$$\sigma_\theta = \frac{EV}{r} + \nu\sigma_x, \quad u_{,x} = \frac{1-\nu^2}{E}\sigma_x - \frac{\nu}{r}V. \tag{6}$$

We next introduce a stress function Φ in the form

$$r\sigma_x = \frac{Ea}{1-\nu^2}(r\Phi)_{,r}, \quad r\tau = \frac{Ea}{1-\nu^2} r\Phi_{,x}, \tag{7}$$

to satisfy the first equation in (1). As τ must vanish at the inner and outer surface, Φ has to satisfy.

$$\Phi(x, a) = \Phi(x, b) = 0. \tag{8}$$

Using eqns (5) and (7) in the second equation in (1) we obtain

$$r\sigma_r = E \ln\frac{r}{a} V + \frac{Ea}{1-\nu^2}\int_a^r \left[\Phi_{,xx} + \frac{\nu}{r^2}\Phi\right] r \, \mathrm{d}r, \tag{9}$$

which satisfies $\sigma_r(a, x) = 0$. In order to satisfy $\sigma_r(b, x) = 0$ we must have

$$(1-\nu^2)\ln\frac{b}{a}\frac{V}{a} + \int_a^b \left[\Phi_{,xx} + \frac{\nu}{r^2}\Phi\right] r \, \mathrm{d}r = 0. \tag{10}$$

Eliminating u between the second equations in (5) and (6) and using (7) we get

$$\left[\frac{1}{r}(r\Phi)_{,r}\right]_{,r} + \alpha^2\Phi_{,xx} + \frac{1}{a}V_{,xx} + \frac{\nu}{ar^2}V = 0, \tag{11}$$

where $\alpha^2 = E/G(1-\nu^2)$.

The integro-differential equations (10) and (11) are to be solved subject to the surface conditions (8) and any one of the four sets of conditions in (3).

SERIES SOLUTIONS OF THE BOUNDARY VALUE PROBLEM

Solutions of the eqns (10) and (11) are sought in the form

$$\Phi = \sum_k A_k \Phi_k(\eta)\, e^{-\lambda_k \xi}, \quad V = a \sum_k A_k V_k\, e^{-\lambda_k \xi}, \tag{12}$$

where A_k are constants to be determined using the edge-conditions and where

$$\xi = x/a, \quad \eta = r/a. \tag{13}$$

For the solutions to vanish as $x \to \infty$ we must have Re $\lambda_k > 0$. In order to satisfy the surface condition (8) we must have

$$\Phi_k(1) = 0, \quad \Phi_k(b/a) = 0. \tag{14}$$

Introducing (12) into (10) and (11) we obtain as equations for V_k and Φ_k,

$$(1-\nu^2)\ln \kappa V_k + \int_1^\kappa \left(\lambda_k^2 + \frac{\nu}{\eta^2}\right)\Phi_k \eta\, d\eta = 0, \tag{15}$$

$$\left[\frac{1}{\eta}(\eta\Phi_k)^{\cdot}\right]^{\cdot} + \alpha^2\lambda_k^2\Phi_k + \left(\lambda_k^2 + \frac{\nu}{\eta^2}\right)V_k = 0, \tag{16}$$

where the dots denote differentiation with respect to η and $\kappa = b/a$.

The edge-conditions (3) may be written as

$$\frac{Ea^2}{1-\nu^2}\sum A_k\Phi_k = M^* f(\eta), \quad a\sum A_k V_k = V^* \tag{17a}$$

$$\frac{Ea^2}{1-\nu^2}\sum A_k\Phi_k = M^* f(\eta), \quad \frac{2Ea}{(1-\nu^2)(\kappa+1)}\sum A_k\lambda_k \int_1^\kappa \Phi_k\eta\, d\eta = Q^*, \tag{17b}$$

$$\sum A_k\lambda_k(\alpha^2\Phi_k + V_k) = -\beta^* g(\eta), \quad a\sum A_k V_k = V^*, \tag{17c}$$

$$\sum A_k\lambda_k(\alpha^2\Phi_k + V_k) = -\beta^* g(\eta), \quad \frac{2Ea}{(1-\nu^2)(\kappa+1)}\sum A_k\lambda_k \int_1^\kappa \Phi_k\eta\, d\eta = Q^*, \tag{17d}$$

where M^* and β^* are the edge-bending moment and the weighted edge-rotation defined in[1], by means of the relations

$$M^* = \int_a^b \bar{\sigma}_x\left[r - \frac{a+b}{2}\right]\frac{2r}{a+b}\, dr = \frac{a^2}{\kappa+1}\int_1^\kappa [2\eta - \kappa - 1]\bar{\sigma}_x\eta\, d\eta \tag{18}$$

$$\beta^* = -\frac{3}{2}\int_a^b \left[1 - \left(\frac{2r-a-b}{b-a}\right)^2\right]\frac{u_{,r}\, dr}{b-a} = \frac{6}{(\kappa-1)^3}\int_1^\kappa [\eta^2 - (\kappa+1)\eta + \kappa]\bar{u}_{,r}\, d\eta \tag{19}$$

and the functions f and g are given by

$$f = \frac{a^2}{M^*\eta}\int_1^\eta \bar{\sigma}_x\eta\, d\eta, \quad g = -\frac{1}{\beta^*}\bar{u}_{,r} \tag{20}$$

Our purpose in introducing M^* and β^* here is to obtain the results in terms of quantities which can be readily compared with the results of the asymptotic analysis[1].

BIORTHOGNAL EIGENFUNCTIONS

The constants A_k may be obtained from the edge conditions (17) through the use of the generalized orthogonality of the eigenfunction pair (Φ_k, V_k). This approach has been utilized previously by Prokopov[3], for plates, by Gusein-Zade[4], and Johnson and Little[5] for elastic semi-infinite strips and by Klemm and Little[2] for isotropic solid cylinders. For our problem the generalized orthogonality can be obtained as follows.

We multiply eqn (15) by V_l and add this to the integral of the product of eqn (16) and Φ_l to obtain

$$\left[(1-\nu^2)\ln\kappa V_k + \int_1^\kappa (\lambda_k^2 + \nu/\eta^2)\Phi_k\eta\, d\eta\right] V_l$$
$$+\int_1^\kappa \left[\left\{\frac{1}{\eta}(\eta\Phi_k)^{\cdot}\right\}^{\cdot} + \alpha^2\lambda_k^2\Phi_k + (\lambda_k^2+\nu/\eta^2)V_k\right]\Phi_l\eta\, d\eta = 0. \tag{21}$$

Integration by parts and the use of eqns (14)–(16) with k replaced by l gives

$$(\lambda_k^2 - \lambda_l^2)\int_1^\kappa [(\alpha^2\Phi_k + V_k)\Phi_l + \Phi_k V_l]\eta\, d\eta = 0. \tag{22}$$

The eigenfunctions (Φ_k, V_k) may be normalized to write

$$\int_1^\kappa [(\alpha^2\Phi_k + V_k)\phi_l + \Phi_k V_l]\eta\, d\eta = \delta_{kl}, \tag{23}$$

where δ_{kl} is the Kronecker delta.

We now obtain explicit expressions for the constants A_k from the edge conditions. From (17a) we have

$$\sum A_k\phi_k = \frac{M^*}{G\alpha^2a^2}f,\quad \sum A_kV_k = \frac{V^*}{a}. \tag{24}$$

From (24) and (23) we find

$$A_k = \frac{M^*}{G\alpha^2a^2}f_k + \frac{V^*}{a}b_k \tag{25}$$

where

$$f_k = \int_1^\kappa f(\alpha^2\Phi_k + V_k)\eta\, d\eta,\quad b_k = \int_1^\kappa \Phi_k\eta\, d\eta \tag{26}$$

Thus, in the mixed edge condition case, (17a) the constants A_k are obtained in a straight forward manner.

Using the expression (25) in the second equation in (17b) gives

$$\frac{M^*}{G\alpha^2a^2}\sum f_k\lambda_kb_k + \frac{V^*}{a}\sum b_k^2\lambda_k = \frac{\kappa+1}{2G\alpha^2a}Q^*. \tag{27}$$

That is,

$$\frac{V^*}{a} = \left[\frac{\kappa+1}{2G\alpha^2a}Q^* - \frac{M^*}{G\alpha^2a^2}\sum f_nb_n\lambda_n\right]\Big/\sum b_n^2\lambda_n. \tag{28}$$

Use of (28) in (25) gives, for the edge condition case (17b)

$$A_k = \frac{M^*}{G\alpha^2a^2}\left[f_k - b_k\frac{\sum f_nb_n\lambda_n}{\sum b_n^2\lambda_n}\right] + \frac{Q^*}{G\alpha^2a}\frac{\kappa+1}{2}\frac{b_k}{\sum b_n^2\lambda_n}. \tag{29}$$

Similarly, in the case of (17c) and (17d) we have

$$A_k\lambda_k = -\beta^*\left[g_k - V_k\frac{\sum g_n V_n/\lambda_n}{\sum V_n^2/\lambda_n}\right] + \frac{V^*}{a}\frac{V_k}{\sum V_n^2/\lambda_n}, \tag{30}$$

and

$$A_k\lambda_k = -\beta^* g_k + \frac{Q^*}{G\alpha^2 a}\frac{\kappa+1}{2}V_k, \tag{31}$$

where we have used the notation

$$g_k = \int_1^\kappa g\Phi_k\eta\,\mathrm{d}\eta. \tag{32}$$

In the case of general transversely isotropic cylinders it is known[2] that the constants A_k have to be obtained by solving an infinite system of equations whenever the boundary condtions are of the form (17b) or (17c). However, in the limiting type transverse isotropy this difficulty does not arise.

INFLUENCE COEFFICIENTS

Given two quantities, one from each one of the two sets (M^*, β^*) and (V^*, Q^*) it is of interest to obtain expressions for the remaining two quantities in terms of these. Corresponding to the edge-condition cases (17a) to (17d) we use generalized influence coefficients G, to write

$$\beta^* = G^{(a)}_{\beta M}M^* + G^{(a)}_{\beta V}V^*,\quad Q^* = G^{(a)}_{QM}M^* + G^{(a)}_{QV}V^*, \tag{33a}$$

$$\beta^* = G^{(b)}_{\beta M}M^* + G^{(b)}_{\beta Q}Q^*,\quad V^* = G^{(b)}_{VM}M^* + G^{(b)}_{VQ}Q^*, \tag{33b}$$

$$M^* = G^{(c)}_{M\beta}\beta^* + G^{(c)}_{MV}V^*,\quad Q^* = G^{(c)}_{Q\beta}\beta^* + G^{(c)}_{QV}V^*, \tag{33c}$$

$$M^* = G^{(d)}_{M\beta}\beta^* + G^{(d)}_{MV}V^*,\quad V^* = G^{(d)}_{V\beta}\beta^* + G^{(d)}_{VQ}Q^*, \tag{33d}$$

where $G_{\beta M}$, $G_{\beta Q}$, G_{VM} and G_{VQ} are flexibility coefficients, $G_{M\beta}$, G_{MV}, $G_{Q\beta}$ and G_{QV} are stiffness coefficients and the remaining coefficients are of a mixed nature. From the defining relations (4), (18) and (19) it is clear that these coefficients are dependent on the edge-stress distribution $f(\eta)$ or on the edge-rotation distribution $g(\eta)$. Furthermore, these coefficients may be expressed in terms of A_k once A_k are known. We, next illustrate this in the case (17b) when $Q^* = 0$ and in the case (17c) when $V^* = 0$. Using

$$u_{,r} = \sum \lambda_k A_k[\alpha^2\Phi_k + V_k] = -\sum \frac{A_k}{\lambda_k}\left\{\left[\frac{1}{\eta}(\eta\Phi_k)^{\cdot}\right]^{\cdot} + \frac{\nu}{\eta^2}V_k\right\}, \tag{34}$$

$$\sigma_x = \frac{G\alpha^2}{\eta}\sum A_k(\eta\Phi_k)^{\cdot}, \tag{35}$$

in eqns (19) and (18) respectively, we have

$$\beta^* = (\kappa+1)\sum\frac{A_k c_k}{\lambda_k},\quad M^* = -\frac{2G\alpha^2a^2}{\kappa+1}\sum A_k b_k, \tag{36}$$

where

$$c_k = \frac{6}{(\kappa-1)^3}\left[\nu\left(\ln\kappa - 2\frac{\kappa+1}{\kappa-1}\right)V_k - \int\frac{\Phi_k}{\eta}\,\mathrm{d}\eta\right]. \tag{37}$$

Introducing (29) and (30) into (36) we obtain

$$G_{\beta M}^{(b)}=\frac{\kappa+1}{G\alpha^2a^2}\sum\left[f_k-b_k\frac{\sum f_n\lambda_nb_n}{\sum\lambda_nb_n^2}\right]\frac{c_k}{\lambda_k} \tag{38}$$

$$G_{M\beta}^{(c)}=\frac{2G\alpha^2a^2}{\kappa+1}\sum\left[g_k-V_k\frac{\sum g_nV_n/\lambda_n}{\sum V_n^2/\lambda_n}\right]\frac{b_k}{\lambda_k}. \tag{39}$$

We note that the flexibility coefficient $G_{\beta M}^{(b)}$ and the stiffness coefficient $G_{M\beta}^{(c)}$ correspond to the coefficients C_{MM} and $K_{\beta\beta}$ used in[6]. However, in[6] the distribution functions f and g were assumed to be a linear function of η and a constant respectively. In elementary shell theory the exact form of the functions f and g does not affect the influence coefficients; the average quantities M^* and β^* determine the nature of the solution. In thin shell approximation we have

$$G_{\beta M}^{(b)}\simeq C_{MM}^{(e)}=\frac{2}{G\alpha^2a^2}\left(\frac{108}{1-\nu^2}\right)^{1/4}\frac{(\kappa+1)^{1/2}}{(\kappa-1)^{5/2}} \tag{40}$$

$$G_{M\beta}^{(c)}\simeq K_{\beta\beta}^{(e)}=2/C_{MM}^{(e)}. \tag{41}$$

We may write eqns (38) and (39) in the form

$$G_{\beta M}^{(b)}=C_{MM}^{(e)}\left(\frac{1-\nu^2}{108}\right)^{1/4}\sqrt{\left(\frac{\kappa+1}{\kappa-1}\right)\frac{(\kappa-1)^3}{2}}\sum\left[f_k-b_k\frac{\sum f_nb_n\lambda_n}{\sum b_n^2\lambda_n}\right]\frac{c_k}{\lambda_k} \tag{42}$$

$$G_{M\beta}^{(c)}=K_{\beta\beta}^{(e)}\left(\frac{108}{1-\nu^2}\right)^{(1/4)}\sqrt{\left(\frac{\kappa-1}{\kappa+1}\right)\frac{2}{(\kappa-1)^3}}\sum\left[g_k-V_k\frac{\sum g_nV_n/\lambda_n}{\sum V_n^2/\lambda_n}\right]\frac{b_k}{\lambda_k} \tag{43}$$

It is expected that $G_{\beta M}^{(e)}\to C_{MM}^{(e)}$ and $G_{M\beta}^{(e)}\to K_{\beta\beta}^{(e)}$ as $\kappa\to1$. For an explicit evaluation to these coefficients we return to the eigenvalue problem expressed by the eqns (15) and (16).

THE EIGENVALUE PROBLEM

Guided by the results of[1] we expect that the eigenvalues (with positive real parts) λ_k consist of a pair of complex conjugate numbers λ_0, $\bar{\lambda}_0$ and a series of real numbers λ_k, $k=1,2,\ldots$; with λ_0, $\bar{\lambda}_0$ being associated with the 'interior' solutions and the remaining λ_k being associated with the 'edge-zone' solutions of the asymptotic theory. Furthermore, we know from[1] that

$$\kappa\to1,\quad \lambda_0,\bar{\lambda}_0\sim\left[\frac{3(1-\nu^2)}{4}\right]^{1/4}\left[\frac{\kappa+1}{\kappa-1}\right]^{1/2}(1\pm i), \tag{44}$$

$$\lambda_k\sim\frac{2k\pi}{\alpha(\kappa-1)}\quad(\text{when } k\gg1). \tag{45}$$

Again, guided by the results of[1] we seek the eigenfunctions in the form

$$\Phi_0=\chi_0+\sum_1 C_{0n}\psi_n,\quad \bar{\Phi}_0=\bar{\chi}_0+\sum_1\bar{C}_{0n}\psi_n, \tag{46}$$

$$\Phi_k=\chi_k+\sum_1 C_{kn}\Psi_n;\quad k=1,2,\ldots\infty, \tag{47}$$

where ψ_n, $n=1,2,\ldots$, are real valued functions, orthonormal in $1<\eta<\kappa$ with a weight function η, which satisfy

$$[(\eta\psi_n)'/\eta]'+\mu_n^2\psi_n=0;\quad n=1,2,\ldots \tag{48}$$

and where χ_k satisfies

$$[(\eta\chi_k)'/\eta]' + (\lambda_k^2 + \nu/\eta^2)V_k = 0; \quad k = 0, 1, 2 \ldots \tag{49}$$

with χ_k and ψ_n vanishing at $\eta = 1$ and at $\eta = \kappa$. The solutions of (48) and (49) are given by

$$\psi_n = B_n[Y_1(\mu_n)J_1(\mu_n\eta) - J_1(\mu_n)Y_1(\mu_n\eta)], \tag{50}$$

$$\chi_k = \frac{1}{\eta}\left\{\left[\frac{1-\eta^3}{3} - \frac{\kappa^3-1}{\kappa^2-1}\frac{1-\eta^2}{3}\right]\lambda_k^2 + \nu(\eta-1)\left(1-\frac{\eta+1}{\kappa+1}\right)\right\}V_k, \tag{51}$$

where the eigenvalues μ_n satisfy $\psi_n(\kappa) = 0$ and where B_n are constants.
In order to have

$$\int_1^\kappa \psi_n\psi_m\eta \, \mathrm{d}\eta = \delta_{nm} \tag{52}$$

we must have

$$B_n = \frac{\pi\mu_n}{2}\frac{J_1(\mu_n\kappa)}{\sqrt{(J_1^2(\mu_n) - J_1^2(\mu_n\kappa))}}. \tag{53}$$

Here, J_1 and Y_1 are the Bessel functions of the first order.

Use of eqn (47) in (16) gives

$$\sum C_{kn}(\alpha^2\lambda_k^2 - \mu_n^2)\psi_n = -\alpha^2\lambda_k^2\chi_k. \tag{54}$$

From the orthonormal property of ψ_n we get

$$\chi_k = V_k\sum\frac{p_n\lambda_k^2 + \nu q_n}{\mu_n^2}\psi_n, \quad C_{kn} = \frac{\alpha^2\lambda_k^2}{\mu_n^2}\frac{p_n\lambda_k^2 + \nu q_n}{\mu_n^2 - \alpha^2\lambda_k^2}V_k \tag{55}$$

where

$$p_n = \int_1^\kappa \psi_n\eta \, \mathrm{d}\eta, \quad q_n = \int_1^\kappa \psi_n\eta^{-1} \, \mathrm{d}\eta. \tag{56}$$

Introducing (50) in (56) we evaluate p_n and q_n in the form.

$$p_n = \frac{2B_n}{\pi\mu_n^2}\left\{\frac{J_1(\mu_n)}{J_1(\mu_n\kappa)}\left[1 + \frac{1}{(\mu_n\kappa)^2} - \frac{1^2.3}{(\mu_n\kappa)^4} + \frac{1^2.3^2.5}{(\mu_n\kappa)^6}\cdots\right] - \left[1 + \frac{1}{\mu_n^2} - \frac{1^2.3}{\mu_n^4} + \frac{1^2.3^2.5}{\mu_n^6} - \cdots\right]\right\}, \tag{57}$$

$$q_n = \mu_n^2 p_n - \frac{2}{\pi}\left\{\frac{J_1(\mu_n)}{J_1(\mu_n\kappa)} - 1\right\}. \tag{58}$$

We next introduce (47) in (15) to get the characteristic equation for λ_k,

$$\frac{\lambda_\kappa^4}{36}\frac{(\kappa-1)^3}{\kappa+1}(\kappa^2+4\kappa+1) + \lambda_k^2\frac{\nu}{3}\frac{(\kappa-1)^3}{\kappa+1} + \ln\kappa - 2\nu^2\frac{\kappa-1}{\kappa+1} + \alpha^2\lambda_k^2\sum\frac{1}{\mu_n^2}\frac{(p_n\lambda_k^2 + \nu q_n)^2}{\mu_n^2 - \alpha^2\lambda_k^2} = 0. \tag{59}$$

The biorthogonality condition (23) determines V_k in the form

$$\frac{1}{V_k^2}=\sum\frac{p_n\lambda_k^2+\nu q_n}{(\mu_n^2-\alpha^2\lambda_k^2)^2}\{2p_n\mu_n^2+\alpha^2\nu q_n-\alpha^2\lambda_k^2p_n\}. \tag{60}$$

With this, for any given edge—stress or edge—rotation distributions it is possible to obtain the solution of our problem. In what follows we consider two special cases of edge conditions.

Linear edge—stress distribution

We consider an edge—stress distribution of the form

$$r\sigma_x=\frac{6(\kappa+1)}{(\kappa-1)}\left[\frac{r}{a}-\frac{\kappa+1}{2}\right]\frac{M^*}{a}. \tag{61}$$

From eqn (20) we then have

$$f(\eta)=\frac{3(\kappa+1)}{(\kappa-1)^3}\left[\eta-(\kappa+1)+\frac{\kappa}{\eta}\right]. \tag{62}$$

Using (62) in (26) we obtain the constants f_k as

$$f_k=\frac{3(\kappa+1)^2}{\lambda_k^2(\kappa-1)^3}\left\{\left(\ln\kappa-2\frac{\kappa-1}{\kappa+1}\right)\nu-\sum\frac{p_n\lambda_n^2+\nu q_n}{\mu_n^2-\alpha^2\lambda_k^2}q_n\right\}V_k. \tag{63}$$

In order to evaluate the influence coefficients $G^b_{\beta M}=C_{MM}$ we have to introduce

$$c_k=\frac{2\lambda_k^2f_k}{(\kappa+1)^2},\quad b_k=\sum\frac{p_n\lambda_k^2+\nu g_n}{\mu_n^2-\alpha^2\lambda_k^2}p_nV_k, \tag{64}$$

in eqn (42).

Constant edge-rotation distribution

Here, we take

$$\bar{u}_{,r}=-\beta^*,\quad \text{i.e.}\quad g(\eta)=1 \tag{65}$$

and with this, from (32) we obtain

$$g_k=b_k=\sum\frac{p_n\lambda_k^2+\nu q_n}{\mu_n^2-\alpha^2\lambda_k^2}p_nV_k \tag{66}$$

which are to be used in eqn (43) to evaluate $G^{(c)}_{M\beta}=K_{\beta\beta}$.

Numerical results. With the objective of comparing the present results with the results of [1] and of the elementary shell theory we consider moderately thick shells with $\kappa=b/a$ having the values 1.1 and 1.25. Table 1 gives the first ten roots of the equation

$$Y_1(\mu)J_1(\mu\kappa)-J_1(\mu)Y_1(\mu\kappa)=0. \tag{67}$$

For large values of n, the nth root μ_n may be obtained using an asymptotic formula given in [7]. The roots of the characteristic eqn (59) when $\nu=1/3$ and $\alpha^2=0.3$, 3 are given in Table 2. We also include the values of λ_k obtained using the asymptotic theory [1] for the case $\kappa=1.25$, $\alpha^2=3$ in parentheses. These two sets of numbers are remarkably close. The odd numbered roots λ_k in parentheses correspond to $2k\pi/\alpha(\kappa-1)$ and the even numbered roots correspond to $2\lambda^*/\alpha(\kappa-1)$ where λ^* satisfies $\tan\lambda^*=\lambda^*$.

The significance of the "interior" solution contributions as compared to the "edge zone" solution contributions is illustrated in Table 3 in terms of the coefficients A_k and V_k entering the

Table 1. Roots of $Y_1(\mu)J_1(\mu\kappa) - J_1(\mu)Y_1(\mu\kappa) = 0$

n	μ_n	
	$\kappa = 1{\cdot}1$	$\kappa = 1{\cdot}25$
1	31·4268	12·5900
2	62·8373	25·1447
3	94·2514	37·7071
4	125·6664	50·2715
5	157·0818	62·8366
6	188·4975	75·4022
7	219·9132	87·9680
8	251·3288	100·5339
9	282·7448	113·1000
10	314·1606	125·6661

Table 2. Roots of eqn (60)

k	$\kappa = 1{\cdot}1$		$\kappa = 1{\cdot}25$	
	$\alpha^2 = 0{\cdot}3$	$\alpha^2 = 3$	$\alpha^2 = 0{\cdot}3$	$\alpha^2 = 3$
0	3·9352	4·0175	2·4006	2·5172 (2·7108)
	±i 3·9552	±i 3·8718	±i 2·4291	±i 2·3088 (2·7108)
1	114·7181	36·2770	45·8934	14·5127 (14·5104)
2	164·0888	51·8871	65·6561	20·7572 (20·7542)
3	229·4311	72·5525	91·7755	29·0220 (29·0207)
4	282·0957	89·2060	112·8498	35·6852 (35·6814)
5	344·1456	108·8284	137·6601	43·5319 (43·5312)
6	398·1711	125·9126	159·2756	50·3670 (50·3640)
7	458·8602	145·1043	183·5454	58·0421 (58·0416)
8	513·6353	162·4256	205·4582	64·9714 (64·9689)
9	573·5747	181·3802	229·4309	72·5524 (72·5520)
10	628·8254	198·8521	251·5330	79·5416 (79·5392)

Table 3. Values of A_k^* and V_k^* when $\kappa = 1.25$, $\alpha^2 = 3$, $\nu = 1/3$

κ	A_k^*	V_k^*
0	±3.9912 + i 0.3908	±3.4102 + i 4.2470
1	0.1152	0.0817
2	0.0280	1.0302
3	0.1450	0.0409
4	0.0048	0.5975
5	0.0046	0.0273

eigenfunction expansions (12). Here, we have taken the linear edge-stress distribution and have introduced the notation

$$A_k, V_k = \frac{1-\nu^2}{Ea^2} M^*(A_k^*, V_k^*). \tag{68}$$

The complex values A_0^*, V_0^* correspond to the interior solution. The remaining coefficients correspond to the edge-zone solution.

Table 4. Influence coefficients $G^{(b)}_{\beta m}$ and $G^{(c)}_{m\beta}$

	$G^{(b)}_{\beta m}/C^{(e)}_{mm}$		$G^{(c)}_{m\beta}/K^{(e)}_{\beta\beta}$	
	$\kappa = 1.1$	$\kappa = 1.25$	$\kappa = 1.1$	$\kappa = 1.25$
$\alpha^2 = 0.3$	1·0257	0·9949	0·9880	0·9711
	(0·9974)	(0·9941)	(0·9881)	(0·9774)
$\alpha^2 = 3{\cdot}0$	1·0197	1·0433	0·9318	0·8560
	(1·0184)	(1·0430)	(0·9252)	(0·8254)
$\alpha^2 = 30{\cdot}0$	1·2069	0·06	0·6307	0·0427

The influence coefficients $G^{(b)}_{\beta M}$ and $G^{(c)}_{M\beta}$ are given in Table 4. The asymptotic theory gives

$$G^{(b)}_{\beta M} = C^{(e)}_{MM}\left[1+\left(\frac{3E}{20G}-\frac{\nu}{4}\right)\frac{\rho}{m^2}+0(\rho^2)\right] \tag{69}$$

$$G^{(c)}_{M\beta} = K^{(e)}_{\beta\beta}\left[1-\left(\frac{9E}{20G}+\frac{\nu}{4}\right)\frac{\rho}{m^2}+0(\rho^2)\right] \tag{70}$$

where $\rho = (\kappa-1)/(\kappa+1)$ and $m^4 = 3(1-\nu^2)/4$. The influence coefficients obtained using (69) and (70) when the terms of order ρ^2 are neglected are shown in parentheses. We note the closeness of the exact coefficients and the asymptotic results. These calculations were performed using 50 terms in the series (12). Sufficient convergence is obtained except in the case of $\alpha^2 = 30.$, $\kappa = 1.25$. When $\alpha^2 = 30$ eqn (70) is inadequate to obtain the value of $G^{(c)}_{M\beta}$, even approximately. In this case the asymptotic values are not presented.

The Limiting case $G\to\infty$. A closed form solution for the present problem may be obtained when $E/G\to 0$, i.e. $\alpha\to 0$. The characteristic eqn (59) for λ, in this case , reduces to

$$\lambda_0^4\frac{(\kappa-1)^3}{36}\frac{\kappa^2+4\kappa+1}{\kappa+1}+\lambda_0^2\frac{\nu}{3}\frac{(\kappa-1)^3}{\kappa+1}+\ln\kappa-2\nu^2\frac{\kappa-1}{\kappa+1}=0. \tag{71}$$

When $\kappa\to 1$ we set $\kappa = 1+\rho(\kappa+1)$ where $\rho \ll 1$ and expand functions of κ to obtain

$$\lambda_0^4+2\nu\lambda_0^2+3(1-\nu^2)/\rho^2=0. \tag{72}$$

This is the characteristic equation corresponding to a shell theory associated with the names of Flügge, Byrne and Lure'. Noting that $\lambda_0^2\sim 0(1/\rho)$, we may neglect the λ_0^2-term in (72) and reduce it to the elementary theory result

$$\lambda_0^4+3(1-\nu^2)/\rho^2=0. \tag{73}$$

The numerical example considered earlier with $\kappa = 1.25$, $\nu = 1/3$ in this limiting case leads to

$$\lambda_0 = 2.7108 \pm i\,2.7108, \tag{74}$$

on the basis of elementary shell theory, and to

$$\lambda_0 = 2.6799 \pm i\,2.7414, \tag{75}$$

on the basis of the FBL theory. The exact result from (72) is given by

$$\lambda_0 = 2.3874 \pm i\,2.4421. \tag{76}$$

The additional values of λ shown in Table 2 represent the effect of shear deformation and the associated St. Venant edge-zone. In the case of vanishing shear deformability, the stress function Φ and the displacement V are given by

$$\Phi = A_0\Phi_0 + \bar{A}_0\Phi_0, \quad \mathrm{V} = \mathrm{a}(\mathrm{V}_0 + \bar{\mathrm{V}}_0) \tag{77}$$

where

$$\Phi_0 = \chi_0 = \frac{1}{\eta}\left\{\frac{1-\eta^3}{3} - \frac{\kappa^3-1}{\kappa^2-1}\frac{1-\eta^2}{3}\lambda_0{}^2 + \nu(\eta-1)\left(1-\frac{\eta+1}{\kappa+1}\right)\right\} V_0 \tag{78}$$

The constants A_0, V_0 (and their complex conjugates) can be obtained for the four cases of edge-conditions. However, the "line" type edge-conditions expressed in terms of $f(\eta)$ and $g(\eta)$ in eqns (17), now affect the solution only through their integrals

$$f_0 = V_0\int_1^\kappa f\eta\, \mathrm{d}\eta, \quad g_0 = \int_1^\kappa g\chi_0\eta\, \mathrm{d}\eta. \tag{79}$$

It is obvious that explicit determination of the influence coefficients can be carriedout in this case.

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